Theories for the Dynamic Response of Laminated Plates

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The effect of the heterogeneous shear deformation over the thickness of the plate on the dynamical behavior of laminated plates is investigated. Three sets of governing equations are derived according to different assumptions on the local transverse shear deformation and the interface conditions. The equations are evaluated by comparing the solutions with the exact solution for harmonic wave propagations. A special formulation for laminates with midplane symmetry is also presented and discussed. It is found that the effect of the local shear deformation depends highly on the transverse shear rigidities of the constituent layers.

1. Introduction

CONSIDERABLE attention has been given to the development of various plate theories for laminates due to the increasing use of fiber-reinforced materials. In the previous works[1-3] the effect of nonhomogeneity of laminates in the thickness direction, such as the bending-extensional coupling in unsymmetrical laminates, has been taken into account to some extent. However, these plate theories essentially followed either the classical Kirchhoff's hypothesis or Mindlin's theory for homogeneous plates, namely, the assumption that plane sections before deformation remain plane after deformation was retained; and the differential deformations in the constituent layers of the laminated plate resulting from the distinct material property neglected. The validity and limitation of such assumption remain to be investigated. In the recent papers,[4,5] it was shown that the plane-section assumption could induce serious errors in the case of a special type of laminate that consists of a large number of alternating layers of two very different materials. Substantial error in the frequency of vibration was found even in the range of rather long wavelength if the stiffnesses of the two materials differ significantly.

In the first part of the paper, we are concerned with the effect of the "heterogeneous shear deformation" over the thickness of the plate on the dynamical behavior of general laminates. Three sets of governing equations are derived by using the energy principle. With the first set of equations, distinct shear deformations in the individual layers are allowed to exist. The local transverse shear deformations according to the second set of equations are constrained in order that the shear stresses be continuous at the interfaces of the layers. In the derivation of the third set of equations we require that the transverse shear deformations in the layers be identical. The third model is identical to that developed by Ref. 3. The three models are then evaluated by comparing their solutions with the solution for the free vibration of a laminated plate according to the exact analysis.

In the second part of the paper, a special class of laminates with midplane symmetry is considered. With a certain assumption regarding the deformation in the plate, a set of equations of motion is derived, which accounts for both the gross and local transverse shear deformations. Vibrations of a three-layered plate are investigated for both cases where the layers are isotropic and orthotropic, respectively.

2. Governing Equations for General Laminates

We consider a laminated plate consisting of a finite number of thin layers of orthotropic materials in perfect bond. The material properties of the laminas may be entirely different. In each layer, the normal stress σ_z is assumed to vanish. Taking this condition into account the constitutive relations in each layer can be expressed in terms of the elastic constants c_{ij} and the reduced stiffnesses Q_{ij} as

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{bmatrix} \tag{1}$$
$$\begin{bmatrix}\sigma_{yz}\\ \sigma_{xz}\end{bmatrix} = \begin{bmatrix}c_{44} & c_{45}\\ c_{45} & c_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_{yz}\\ \varepsilon_{xz}\end{bmatrix}$$

where ε_{xy}, ε_{yz} and ε_{xz} are engineering strains.

Consider the kth layer of the plate. The displacement field in this layer is assumed to be of the form

$$\begin{aligned}\bar{u}^{(k)} &= u^{(k)}(x,y,t) - \bar{z}^{(k)}\phi_x{}^{(k)}(x,y,t)\\ \bar{v}^{(k)} &= v^{(k)}(x,y,t) - \bar{z}^{(k)}\phi_y{}^{(k)}(x,y,t)\\ \bar{w}^{(k)} &= w^{(k)}(x,y,t) = w(x,y,t)\end{aligned} \tag{2}$$

where $\bar{z}^{(k)}$ is the thickness-coordinate with the origin loated at the midplane of the kth layer; $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ are the displacements at the midplane of the kth layer in the x-, y- and z-directions. It is also assumed in Eqs. (2) that the transverse displacements in the layers are equal, i.e., $w^{(k)} = w$ for all k.

The approximate expression for the displacement in each layer can be used to calculate the strain components. From Eqs. (1), the stress components in the individual layers can be derived. The strain energy per unit area stored in the kth layer is obtained as

$$U^{(k)} = \frac{1}{2}\int_{-d_{(k)}/2}^{d_{(k)}/2} [\sigma_x{}^{(k)}\varepsilon_x{}^{(k)} + \sigma_y{}^{(k)}\varepsilon_y{}^{(k)} + \sigma_{xy}{}^{(k)}\varepsilon_{xy}{}^{(k)} + \kappa\sigma_{yz}\varepsilon_{yz}{}^{(k)} + \kappa\sigma_{xz}{}^{(k)}\varepsilon_{xz}{}^{(k)}]\, d\bar{z}^{(k)} \tag{3}$$

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In Eq. (3), $d_{(k)}$ is the thickness of the kth layer, and κ is the shear correction coefficient. The local strain energy function $U^{(k)}$ can be expressed in terms of the displacement components at the midplane of the kth layer, $u^{(k)}$, $v^{(k)}$ and $w(=w^{(k)})$, and the rotations $\phi_x^{(k)}$ and $\phi_y^{(k)}$. We have

$$\begin{aligned}2U^{(k)} = {} & d_{(k)}Q_{11}^{(k)}(u_{,x}^{(k)})^2+\tfrac{1}{12}d_{(k)}^3Q_{11}^{(k)}(\phi_{x,x}^{(k)})^2+ \\ & 2d_{(k)}Q_{12}^{(k)}(u_{,x}^{(k)})(v_{,y}^{(k)})+ \\ & \tfrac{1}{6}d_{(k)}^3Q_{12}^{(k)}(\phi_{,x}^{(k)})(\phi_{,y}^{(k)})+d_{(k)}Q_{22}^{(k)}(v_{,y}^{(k)})^2+ \\ & \tfrac{1}{12}d_{(k)}^3Q_{22}^{(k)}(\phi_{y,y}^{(k)})^2+2d_{(k)}Q_{16}^{(k)}(u_{,x}^{(k)})\times \\ & (u_{,y}^{(k)}+v_{,x}^{(k)})+\tfrac{1}{6}d_{(k)}^3Q_{16}^{(k)}(\phi_{x,x}^{(k)})(\phi_{x,y}^{(k)}+\phi_{y,x}^{(k)})+ \\ & 2d_{(k)}Q_{26}^{(k)}(v_{,y}^{(k)})(u_{,y}^{(k)}+v_{,x}^{(k)})+ \\ & \tfrac{1}{6}d_{(k)}^3Q_{26}^{(k)}(\phi_{,y}^{(k)})(\phi_{x,y}^{(k)}+\phi_{y,x}^{(k)})+ \\ & d_{(k)}Q_{66}^{(k)}(u_{,y}^{(k)}+v_{,x}^{(k)})^2+ \\ & \tfrac{1}{12}d_{(k)}^3Q_{66}^{(k)}(\phi_{x,y}^{(k)}+\phi_{y,x}^{(k)})^2+ \\ & 2d_{(k)}\kappa c_{44}^{(k)}(w_{,y}-\phi_y^{(k)})^2+ \\ & 2d_{(k)}\kappa c_{45}^{(k)}(w_{,x}-\phi_x^{(k)})(w_{,y}-\phi_{,y}^{(k)})+ \\ & d_{(k)}\kappa c_{55}^{(k)}(w_{,x}-\phi_x^{(k)})^2\end{aligned} \tag{4}$$

where the differentiation is denoted by a comma.

The total strain energy per unit area of the laminated plate is given by

$$U=\sum_k U^{(k)} \tag{5}$$

A similar expression for the plate-kinetic-energy for the laminated plate based upon Eqs. (2) is obtained as

$$T=\sum_k T^{(k)} \tag{6}$$

where

$$T^{(k)}=\tfrac{1}{2}\rho^{(k)}d_{(k)}[(\dot{u}^{(k)})^2+(\dot{v}^{(k)})^2+(\dot{w})^2]+\tfrac{1}{24}\rho^{(k)}d_{(k)}^3[(\dot{\phi}_x^{(k)})^2+(\dot{\phi}_y^{(k)})^2] \tag{7}$$

In Eq. (7), $\rho^{(k)}$ is the mass density and a dot indicates differentiation with respect to time.

With the plate-strain-energy function, Eq. (4), and the plate-kinetic-energy function, Eq. (6), obtained, the natural way to derive the equations of motion is by the use of Hamilton's principle. However, we note that there are certain constraint conditions at the interface of the layers to satisfy, and, as a consequence, not all the displacement components and rotations in the layers are independent kinematical quantities. These constraint conditions must be taken into account before the variational principle is employed. Depending on various constraint conditions, we obtain various theories for the laminated plate.

Theory I

It is assumed that the layers in the plate are perfectly bonded. If we require that the displacement given by Eqs. (2) be continuous at the interface of the layers, we obtain the relations

$$\begin{aligned}u^{(k+1)}-u^{(k)}&=\tfrac{1}{2}d_{(k)}\phi_x^{(k)}+\tfrac{1}{2}d_{(k+1)}\phi_x^{(k+1)}\\ v^{(k+1)}-v^{(k)}&=\tfrac{1}{2}d_{(k)}\phi_y^{(k)}+\tfrac{1}{2}d_{(k+1)}\phi_y^{(k+1)}\end{aligned} \tag{8}$$

Using Eqs. (8), $u^{(k+1)}$ and $v^{(k+1)}$ can be eliminated from Eqs. (5) and (6). Consequently, the plate-strain-energy function U and the plate-kinetic-energy function T can be written in terms of a pair of in-plane displacements, say $u^{(1)}$ and $v^{(1)}$, the transverse displacement w and the local rotations $\phi_x^{(k)}$ and $\phi_y^{(k)}$.

Consider a plane region A of the laminated plate. The displacement equations of motion can be obtained by invoking Hamilton's principle for variations of the independent kinematic variables in A and any time interval $t_0 \leqq t \leqq t_1$, i.e.,

$$\delta\int_{t_0}^{t_1}\int_A (T-U)\,dt\,dA+\int_{t_0}^{t_1}\delta W_1\,dt=0 \tag{9}$$

where δW_1 is the variation of the work done by external forces, and dA is the scalar area element. Since we are presently interested only in the displacement equations of motion, we restrict the admissible variations to the ones that vanish identically on the boundary contour of A. In the absence of body forces the variational problem reduces to finding the Euler equations for

$$\delta\int_{t_0}^{t_1}\int_A F\,dt\,dA=0 \tag{10}$$

where the functional F is defined as

$$F=T-U \tag{11}$$

For the laminated plate that consists of two layers, the equations of motion for the case of plane-strain parallel to the x-z plane are obtained as

$$\begin{aligned}&a_1u_{,xx}^{(1)}+a_2\phi_{x,xx}^{(1)}+a_3\phi_{x,xx}^{(2)}=a_4\ddot{u}^{(1)}+a_5\ddot{\phi}_x^{(1)}+a_6\ddot{\phi}_x^{(2)}\\ &a_7w_{,xx}+a_8\phi_{x,x}^{(1)}+a_9\phi_{x,x}^{(2)}=a_{10}\ddot{w}\\ &a_{11}u_{1,xx}^{(1)}+a_{12}w_{,x}+a_{13}\phi_{x,xx}^{(1)}+a_{14}\phi_x^{(1)}+a_{15}\phi_{x,xx}^{(2)}=a_{16}\ddot{u}^{(1)}+a_{17}\ddot{\phi}_x^{(1)}+a_{18}\ddot{\phi}_x^{(2)}\\ &a_{19}u_{,xx}^{(1)}+a_{20}w_{,x}+a_{21}\phi_{x,xx}^{(1)}+a_{22}\phi_{x,xx}^{(2)}+a_{23}\phi_x^{(2)}=a_{24}\ddot{u}^{(1)}+a_{25}\ddot{\phi}_x^{(1)}+a_{26}\ddot{\phi}_x^{(2)}\end{aligned} \tag{12}$$

where the coefficients $a_i(i=1-26)$ are expressed in terms of the elastic constants and the geometrical dimensions of the layers.

Theory II

If, besides the displacement-continuity conditions given by Eqs. (8), we further require that the shear stresses at the interfaces of the layers be continuous, we obtain the additional relations among the kinematical variables. We have

$$\begin{aligned}&c_{45}^{(k)}(w_{,y}-\phi_y^{(k)})+c_{55}^{(k)}(w_{,x}-\phi_x^{(k)})=c_{45}^{(k+1)}(w_{,y}-\phi_y^{(k+1)})+c_{55}^{(k+1)}(w_{,x}-\phi_x^{(k+1)})\\ &c_{44}^{(k)}(w_{,y}-\phi_y^{(k)})+c_{45}^{(k)}(w_{,x}-\phi_x^{(k)})=c_{44}^{(k+1)}(w_{,y}-\phi_y^{(k+1)})+c_{45}^{(k+1)}(w_{,x}-\phi_x^{(k+1)})\end{aligned} \tag{13}$$

In view of Eqs. (13), we are able to reduce further the number of independent kinematic variables. The result is that U and T can then be expressed in terms of w and, say, $u^{(1)}$, $v^{(1)}$, $\phi_x^{(1)}$ and $\phi_y^{(1)}$ for plates consisting of any number of layers. By use of Hamilton's principle we obtain five equations of motion. Since the manipulation involved is very straightforward, the explicit expression of the equations will not be given here.

Theory III

Theory I developed previously can be simplified by making the additional assumption that the local rotations in the individual layers are identical, i.e.

$$\phi_x^{(k)}=\psi_x,\quad \phi_y^{(k)}=\psi_y\quad \text{for all } k \tag{14}$$

Equation (14) simply indicates that plane sections of the plate before deformation remain plane after deformation. It is noted here that this assumption was taken by Refs. 2 and 3. Such a simplification may incur loss of accuracy of the plate theory. It is one of the objectives of the present paper to gain some knowledge of the limitation of the theory developed based upon this assumption.

Shear Correction Coefficients

The determination of the shear correction coefficients is a controversy in the literature. Various values of the shear correction coefficients for homogeneous and isotropic plates have been suggested by Reissner,[6] Mindlin[7] and Uflyand.[8] In the present formulation, each constituent layer behaves as a Mindlin plate. Thus, the shear correction coefficient for each layer can be determined by matching the frequencies of the thickness-shear mode vibration of each layer as determined from the Mindlin plate theory and the exact dynamic elasticity analysis. If the layers are homogeneous and isotropic, then the coefficient assumes the value of $\pi^2/12$. It can also be shown easily that this is also the value for a homogeneous orthotropic plate.

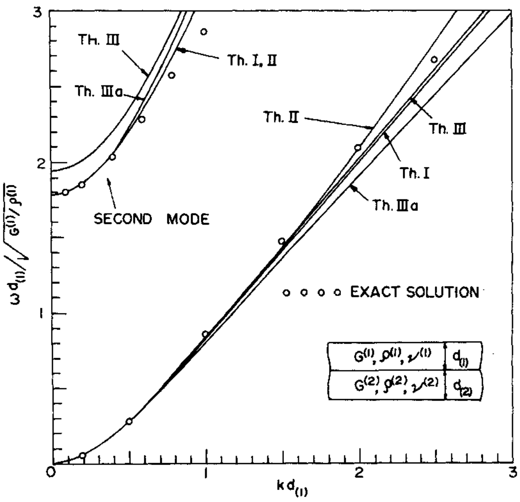

Fig. 1 Frequencies of the two lowest flexural modes in a two-layered plate with $G^{(2)}/G^{(1)} = 2$, $\rho^{(2)}/\rho^{(1)} = 1$, $d_{(2)}/d_{(1)} = 1$. $\nu^{(1)} = \nu^{(2)} = 0.5$.

Yang, Norris and Stavsky[2] suggested another way of determining the shear correction coefficients that applies to Theory III. The procedure involves comparing the frequencies of the thickness-shear mode vibration obtained according to the plate equations of Theory III and the exact elasticity analysis for the laminated plate. The procedure becomes quite tedious once the number of layers is large. Theory III with such shear correction coefficients will be referred to as Theory IIIa.

Comparison with Exact Analysis

In order to provide some evaluation examples for the theories developed, we now consider a special laminated plate consisting of two layers of homogeneous, elastic and isotropic materials. The exact analysis for the one-way harmonic vibration of the two-layered plate of infinite extent has been carried out independently by Jones,[9] and Yang et al.[2] The harmonic vibration according to Theory I is described by

$$
\begin{aligned}
u^{(1)} &= U^* e^{i(kx-\omega t)} \\
w &= W^* e^{i(kx-\omega t)} \\
\phi_x^{(1)} &= \Phi_1^* e^{i(kx-\omega t)} \\
\phi_x^{(2)} &= \Phi_2^* e^{i(kx-\omega t)}
\end{aligned}
\tag{15}
$$

where k is the angular wave number, ω the natural frequency and U^*, W^*, Φ_1^* and Φ_2^* are four constants. Substituting Eqs. (15) in Eqs. (12), we obtain a system of four homogeneous equations for the four constants. The frequency equation is then obtained by equating the determinant of coefficients to zero. It must be noted here that for isotropic materials, we have

$$
\begin{aligned}
Q_{11}^{(k)} &= 2G^{(k)}/(1-\nu^{(k)}) \\
c_{55}^{(k)} &= G^{(k)}
\end{aligned}
\qquad k = 1,2
\tag{16}
$$

where $G^{(k)}$ and $\nu^{(k)}$ are the shear modulus and the Poisson's ratio for the kth layer, respectively. The frequency equations based upon Theory II, Theory III, and Theory IIIa can be derived in the same manner.

Numerical results are shown in Fig. 1 and Fig. 2 for the two lowest flexural modes of vibration with the dimensionless frequency plotted vs the dimensionless wave number. Figure 1 shows the case where the stiffnesses of the two materials are similar. In this case, Theories I and II yield very good agreement with the exact solution for both modes. Results obtained from Theory III are good for the first mode but relatively poor for the second flexural mode. These discrepancies yielded by Theories III and IIIa become very pronounced if the ratio of the shear rigidities of the two materials is large as can be seen from Fig. 2.

The two numerical examples reveal that the assumption that plane sections remain plane after deformation is acceptable in the case where the difference in the shear rigidities of the constituent layers is small. The discrepancy induced by neglecting the local rotations of the individual layers cannot be corrected by shear correction coefficients if the difference in shear rigidity is substantial. For such laminates, the heterogeneous shear deformation in the thickness direction should be incorporated.

3. Midplane Symmetric Laminated Plates

Laminated plates which possess the midplane symmetry form an important class of plates. We will consider a special class of such laminates, which are formed by layers of two different materials. The two types of layers, which will be denoted by Materials "1" and "2," respectively, are stacked alternately, and the layers of the same materials are assumed to have the same thicknesses d_1 and d_2, respectively.

To allow the local rotations, the approximate displacement field for each constituent layer given by Eqs. (2) is again employed here. We have for the kth layer of Material $\alpha(\alpha = 1, 2)$

$$
\begin{aligned}
\bar{u}_\alpha^{(k)} &= u_\alpha^{(k)}(x,y,t) - \bar{z}_\alpha^{(k)}\phi_{x\alpha}^{(k)}(x,y,t) \\
\bar{v}_\alpha^{(k)} &= v_\alpha^{(k)}(x,y,t) - \bar{z}_\alpha^{(k)}\phi_{y\alpha}^{(k)}(x,y,t) \\
\bar{w}_\alpha^{(k)} &= w(x,y,t)
\end{aligned}
\tag{17}
$$

In view of the symmetric nature of the plate, we may, in addition to Eqs. (17), assume that the average displacements of the individual layers, that is, the displacements of the midplanes of the layers, remain in a plane after deformation. The rotations of the constituent layers are then allowed to deviate from the rotation of this plane to account for the heterogeneous characteristics of the laminated plate in the thickness-direction.

Suppose that the laminated plate undergoes a deformation which is flexural in nature. The state of flexural deformation, which is characterized by the vanishing of the in-plane displacement components in the midplane of the plate, is allowed by the symmetric lamination. With the foregoing in mind, we can write the displacement components in the midplane of the kth layer as

$$
\begin{aligned}
u_\alpha^{(k)} &= -z_\alpha^{(k)}\psi_x(x,y,t) \\
v_\alpha^{(k)} &= -z_\alpha^{(k)}\psi_y(x,y,t) \qquad \alpha = 1,2 \\
w_\alpha^{(k)} &= w(x,y,t)
\end{aligned}
\tag{18}
$$

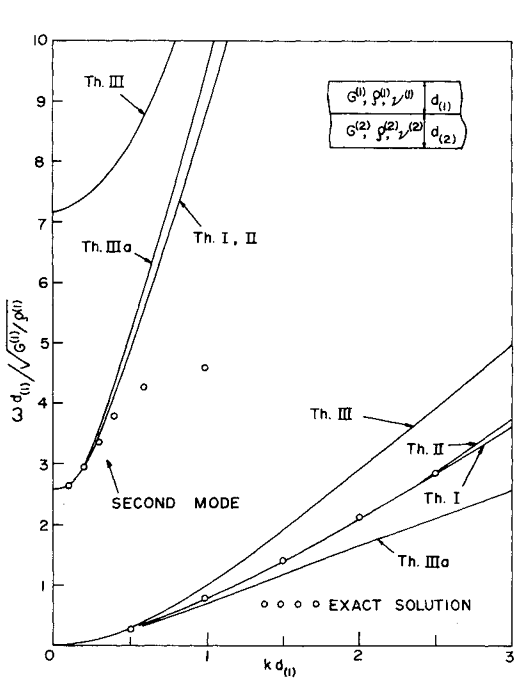

Fig. 2 Frequencies of the two lowest flexural modes for a two-layered plate with $G^{(2)}/G^{(1)} = 40$, $\rho^{(2)}/\rho^{(1)} = 1$, $d_{(2)}/d_{(1)} = 1$, $\nu^{(1)} = \nu^{(2)} = 0.5$.

where $z_\alpha^{(k)}$ denote the positions of the midplanes of the layers of Material α, and ψ_x and ψ_y indicate the rotations of the planes passing through the midplanes of the layers. It is noted that ψ_x and ψ_y may be regarded as the gross rotations of the laminated plate.

In view of Eqs. (17) and (18) and that the displacements are continuous at the interfaces of the layers, we require that $\phi_{x\alpha}^{(k)} = \phi_{x\alpha}(x, y, t)$ and $\phi_{y\alpha}^{(k)} = \phi_{y\alpha}(x, y, t)$ for all k. The conditions given by Eqs. (8) now assume the form

$$\begin{aligned} \psi_x &= \eta\phi_{x1} + (1-\eta)\phi_{x2} \\ \psi_y &= \eta\phi_{y1} + (1-\eta)\phi_{y2} \end{aligned} \tag{19}$$

where

$$\eta = d_1/(d_1+d_2) \tag{20}$$

The strains in the layers can be computed by using the expressions given by Eqs. (17) and (18). Following the same procedure in the previous section, the plate-strain-energy function for the plate consisting of m layers of Material 1 and $n(=m\pm 1)$ layers of Material 2 is derived as

$$\begin{aligned} 2U = {} & b_1(\psi_{x,x})^2 + b_2(\phi_{x1,x})^2 + b_3(\phi_{x2,x})^2 + b_4(\psi_{y,y})^2 + \\ & b_5(\phi_{y1,y})^2 + b_6(\phi_{y2,y})^2 + 2b_7(\psi_{x,x})(\psi_{y,y}) + \\ & 2b_8(\phi_{x1,x})(\phi_{y1,y}) + 2b_9(\phi_{x2,x})(\phi_{y2,y}) + \\ & 2b_{10}(\psi_{x,x})(\psi_{x,y}+\psi_{y,x}) + 2b_{11}(\phi_{x1,x})(\phi_{x1,y}+\phi_{y1,x}) + \\ & 2b_{12}(\phi_{x2,x})(\phi_{x2,y}+\phi_{y2,x}) + 2b_{13}(\psi_{y,y})(\psi_{x,y}+\psi_{y,x}) + \\ & 2b_{14}(\phi_{y1,y})(\phi_{x1,y}+\phi_{y1,x}) + \\ & 2b_{15}(\phi_{y2,y})(\phi_{x2,y}+\phi_{y2,x}) + b_{16}(\psi_{x,y}+\psi_{y,x})^2 + \\ & b_{17}(\phi_{x1,y}+\phi_{y1,x})^2 + b_{18}(\phi_{x2,y}+\phi_{y2,x})^2 + \\ & b_{19}(w_{,y}-\phi_{y1})^2 + b_{20}(w_{,y}-\phi_{y2})^2 + \\ & 2b_{21}(w_{,y}-\phi_{y1})(w_{,x}-\phi_{x1}) + 2b_{22}(w_{,y}-\phi_{y2})(w_{,x}-\phi_{x2}) + \\ & b_{23}(w_{,x}-\phi_{x1})^2 + b_{24}(w_{,x}-\phi_{x2})^2 \end{aligned} \tag{21}$$

where

$$\begin{aligned} b_1 &= Q_{11}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{11}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_2 &= mQ_{11}^{(1)}I_1, \quad b_3 = nQ_{11}^{(2)}I_2 \\ b_4 &= Q_{22}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{22}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_5 &= mQ_{22}^{(1)}I_1, \quad b_6 = nQ_{22}^{(2)}I_2 \\ b_7 &= Q_{12}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{12}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_8 &= mQ_{12}^{(1)}I_1, \quad b_9 = nQ_{12}^{(2)}I_2 \\ b_{10} &= Q_{16}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{16}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_{11} &= mQ_{16}^{(1)}I_1, \quad b_{12} = nQ_{16}^{(2)}I_2 \\ b_{13} &= Q_{26}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{26}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_{14} &= mQ_{26}^{(1)}I_1, \quad b_{15} = nQ_{26}^{(2)}I_2 \\ b_{16} &= Q_{66}^{(1)}d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + Q_{66}^{(2)}d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_{17} &= mQ_{66}^{(1)}I_1, \quad b_{18} = nQ_{66}^{(2)}I_2 \\ b_{19} &= mc_{44}^{(1)}d_1\kappa, \quad b_{20} = nc_{44}^{(2)}d_2\kappa \\ b_{21} &= mc_{45}^{(1)}d_1\kappa, \quad b_{22} = nc_{45}^{(2)}d_2\kappa \\ b_{23} &= mc_{55}^{(1)}d_1\kappa, \quad b_{24} = nc_{55}^{(2)}d_2\kappa \end{aligned} \tag{22}$$

In Eqs. (22), $Q_{ij}^{(\alpha)}$ and $c_{ij}^{(\alpha)}$ are the material constants for Material $\alpha(\alpha = 1, 2)$, and

$$I_\alpha = d_\alpha^3/12 \tag{23}$$

The assumption given by Eqs. (18) and (19) has been employed to describe the incremental deformation in an orthotropic laminated plate under initial stress.[5] However, in Ref. 5, it was assumed that the number of layers was large, so that a smoothing operation could be used. In the present case, such restriction is removed.

The plate-kinetic-energy function is obtained as

$$\begin{aligned} 2T = {} & b_{25}[(\dot\psi_x)^2 + (\dot\psi_y)^2] + b_{26}[(\dot\phi_{x1})^2 + (\dot\phi_{y1})^2] + \\ & b_{27}[(\dot\phi_{x2})^2 + (\dot\phi_{y2})^2] + b_{28}(\dot w)^2 \end{aligned} \tag{24}$$

where

$$\begin{aligned} b_{25} &= \rho_1 d_1\sum_{k=1}^{m}(z_1^{(k)})^2 + \rho_2 d_2\sum_{k=1}^{n}(z_2^{(k)})^2 \\ b_{26} &= m\rho_1 I_1, \quad b_{27} = n\rho_2 I_2 \\ b_{28} &= m\rho_1 d_1 + n\rho_2 d_2 \end{aligned} \tag{25}$$

where ρ_α is the mass density for Material α.

In order to apply Hamilton's principle, we may use the constraint conditions given by Eqs. (19) to eliminate two dependent kinematic variables. An alternative way of taking the continuity conditions into account is to introduce them as subsidiary conditions through the use of Lagrangian multipliers. The functional is then redefined as

$$F = T - U - \Gamma_x S_x - \Gamma_y S_y, \tag{26}$$

where Γ_x and Γ_y are the Lagrangian multipliers, and

$$\begin{aligned} S_x &= \psi_x - \eta\phi_{x1} - (1-\eta)\phi_{x2} \\ S_y &= \psi_y - \eta\phi_{y1} - (1-\eta)\phi_{y2} \end{aligned} \tag{27}$$

In the state of plane-strain parallel to x-z-plane, the equations of motion are obtained as

$$\begin{aligned} & (b_{23}+b_{24})w_{,xx} - b_{23}\phi_{x1,x} - b_{24}\phi_{x2,x} = b_{28}\ddot w \\ & b_1\psi_{x,xx} - \Gamma_x = b_{25}\ddot\psi_x \\ & b_{23}w_{,x} + b_2\phi_{x1,xx} - b_{23}\phi_{x1} + \eta\Gamma_x = b_{26}\ddot\phi_{x1} \\ & b_{24}w_{,x} + b_3\phi_{x2,xx} - b_{24}\phi_{x2} + (1-\eta)\Gamma_x = b_{27}\ddot\phi_{x2} \\ & \psi_x - \eta\phi_{x1} - (1-\eta)\phi_{x2} = 0 \end{aligned} \tag{28}$$

A set of simplified equations can be obtained if we assume that the local rotations are identical to the gross rotations, namely,

$$\begin{aligned} \phi_{x1} &= \phi_{x2} = \psi_x \\ \phi_{y1} &= \phi_{y2} = \psi_y. \end{aligned} \tag{29}$$

Then the plate-strain-energy function simplifies to

$$\begin{aligned} 2U = {} & B_1(\psi_{x,x})^2 + B_2(\psi_{y,y})^2 + B_3(\psi_{x,x})(\psi_{y,y}) + \\ & B_4(\psi_{x,x})(\psi_{x,y}+\psi_{y,x}) + B_5(\psi_{y,y})(\psi_{x,y}+\psi_{y,x}) \\ & B_6(\psi_{x,y}+\psi_{y,x})^2 + B_7(w_{,y}-\psi_y)^2 \quad + \\ & B_8(w_{,y}-\psi_y)(w_{,x}-\psi_x) + B_9(w_{,x}-\psi_x)^2 \end{aligned} \tag{30}$$

where

$$\begin{aligned} & B_1 = b_1+b_2+b_3, \quad B_2 = b_4+b_5+b_6 \\ & B_3 = 2(b_7+b_8+b_9), \quad B_4 = 2(b_{10}+b_{11}+b_{12}) \\ & B_5 = 2(b_{13}+b_{14}+b_{15}), \quad B_6 = b_{16}+b_{17}+b_{18} \\ & B_7 = b_{19}+b_{20}, \quad B_8 = 2(b_{21}+b_{22}) \\ & B_9 = b_{23}+b_{24} \end{aligned} \tag{31}$$

The plate-kinetic-energy function can be written as

$$2T = B_{10}[(\dot\psi_x)^2 + (\dot\psi_y)^2] + B_{11}(\dot w)^2 \tag{32}$$

where

$$\begin{aligned} B_{10} &= b_{25}+b_{26}+b_{27} \\ B_{11} &= b_{28} \end{aligned} \tag{33}$$

It is noted that if the assumption given by Eqs. (29) is taken, then the continuity conditions, Eqs. (19), are identically satisfied. The displacement-equations of motion can then be derived by employing Hamilton's principle. It can be easily verified that the governing equations thus derived are identical to those of Ref. 3 or Theory III in the previous section.

Comparison with Exact Analysis for Isotropic Layers

As an example, we consider a three-layered plate symmetrically stacked with $m = 2$ and $n = 1$. Plane harmonic waves governed by Eqs. (28) are expressed in the form

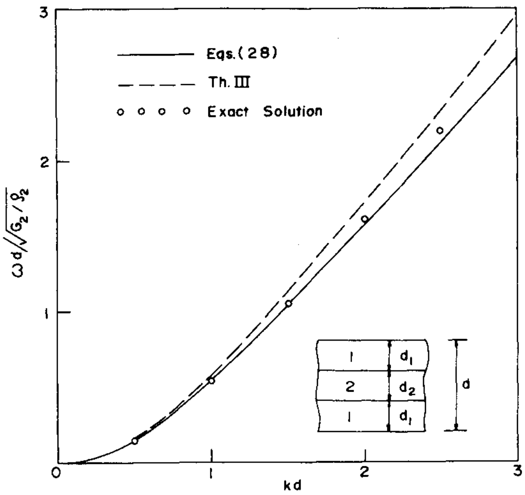

Fig. 3 Fundamental frequencies for a three-layered plate of two isotropic materials ($G_1/G_2 = 2$, $\rho_1/\rho_2 = 1$, $d_1/d_2 = 1$, $v_1 = 0.25$, $v_2 = 0.3$).

$$
\begin{aligned}
w &= W^* e^{i(kx-\omega t)} \\
\psi_x &= \Psi_x^* e^{i(kx-\omega t)} \\
\phi_{x1} &= \Phi_{x1}^* e^{i(kx-\omega t)} \\
\phi_{x2} &= \Phi_{x2}^* e^{i(kx-\omega t)} \\
\Gamma_x &= \Gamma_x^* e^{i(kx-\omega t)}
\end{aligned} \tag{34}
$$

where W^*, Ψ_x^*, Φ_{x1}^*, Φ_{x2}^* and Γ_x^* are constants. The dispersion equation is obtained in the usual manner by substituting Eqs. (34) in Eqs. (28) and requiring the matrix of coefficients of the resulting equations to be singular. The dispersion equation based upon the assumption given by Eqs. (29) can also be derived easily.

If Material 1 and 2 are both isotropic, then we have the relations

$$
\begin{aligned}
Q_{11}^{(\alpha)} &= 2G_\alpha/(1-v_\alpha) \\
c_{55}^{(\alpha)} &= G_\alpha
\end{aligned} \qquad \alpha = 1,2 \tag{35}
$$

where G_α and v_α are the shear modulus and the Poisson's ratio for Material α, respectively.

In Figs. 3 and 4, the dispersion curves for the lowest flexural mode are computed for the three-layered plate with isotropic constituents. The exact solutions are also shown for comparison. As before, we find that the effect of the local shear deformation is small if the ratio of the shear rigidities of the two materials is small, and the frequencies predicted by the simplified theory (or Theory III) agree fairly well with the exact frequencies. In the case of large differences in shear rigidity of the layers, the effect of the local transverse shear deformation can no longer be neglected as evidenced by Fig. 4. It is noticed that the dispersion curves obtained from Eqs. (28) agree very well with the exact solutions for both cases over a wide range of wavelengths.

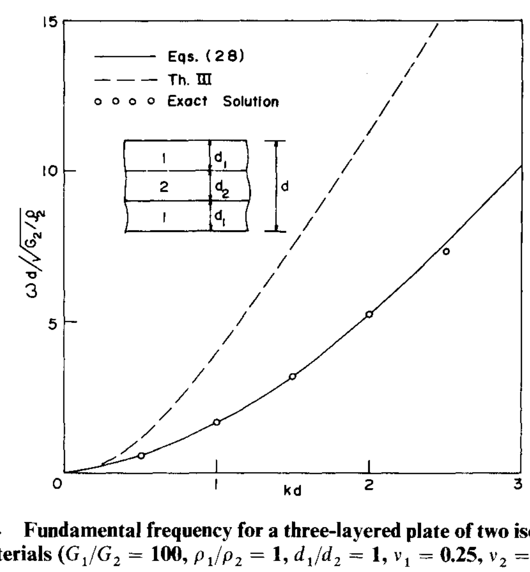

Fig. 4 Fundamental frequency for a three-layered plate of two isotropic materials ($G_1/G_2 = 100$, $\rho_1/\rho_2 = 1$, $d_1/d_2 = 1$, $v_1 = 0.25$, $v_2 = 0.3$).

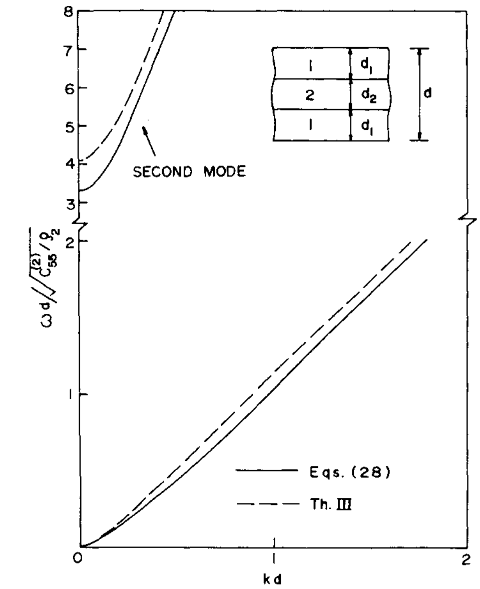

Fig. 5 Frequencies of the two lowest flexural modes for a three-layered plate with orthotropic layers of similar shear rigidities given by Eqs. (36).

Orthotropic Layers

Figures 5 and 6 present the frequencies for the two lowest flexural modes for a three-layered plate with orthotropic layers. The layer properties are given by

$$
c_{55}^{(1)}/c_{55}^{(2)} = 2, \quad Q_{11}^{(2)}/c_{55}^{(2)} = 5, \quad Q_{11}^{(1)}/c_{55}^{(2)} = 250 \\
\rho_1/\rho_2 = 1, \quad d_1/d_2 = 1 \tag{36}
$$

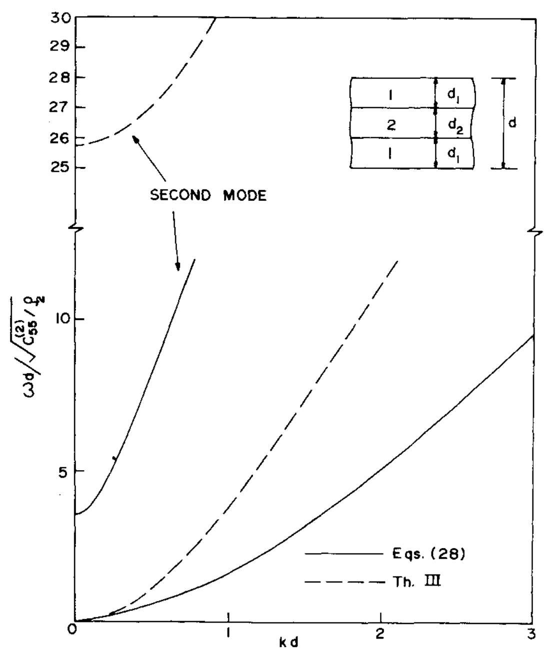

Fig. 6 Frequencies of the two lowest flexural modes for a three-layered plate with orthotropic layers of highly distinct shear rigidities given by Eqs. (37).

and

$$c_{55}^{(1)}/c_{55}^{(2)} = 100, \quad Q_{11}^{(2)}/c_{55}^{(2)} = 5, \quad Q_{11}^{(1)}/c_{55}^{(2)} = 250$$
$$\rho_1/\rho_2 = 1, \quad d_1/d_2 = 1 \tag{37}$$

respectively. It is noted that, in both cases, the longitudinal stiffnesses remain unchanged, while the shear rigidities are assumed to be similar in the first case and substantially different in the second case. Again, we find that if the difference in the shear rigidity is small, the influence of the local transverse shear deformation on the frequency of the fundamental flexural mode is negligible for practical interest. The influence, however, is more pronounced on the second mode. It is shown in Fig. 6 that, once the ratio of the shear rigidities of the two materials is large, the discrepancy between the two theories with and without local rotations becomes very serious, especially in the second mode.

4. Conclusions

In the first part of the paper, three theories for general laminated plates are developed based upon different assumptions on the local transverse shear deformation and the interface conditions. In the second part, a special formulation for laminates with midplane symmetry is presented. Dispersion curves for plane harmonic wave propagations are obtained from the plate theories and compared with the curves obtained from the exact elasticity analysis. It is found from the numerical examples that the theories which incorporate local deformations, in general, yield results in good agreement with the exact solution. The theories assuming that plane sections before deformation remain plane after deformation are accurate in predicting the fundamental mode of vibration for laminates with similar transverse shear rigidities of the constituent layers. For laminates with large ratio of shear moduli, the foregoing assumption is found inadequate. It should also be noted that the theories which include local rotations account for more modes of vibration than those which do not. However, if the primary interest is to develop plate theories for laminates consisting of layers of unidirectionally fiber-reinforced materials stacked in different orientations, the effect of the local shear deformation may be neglected in the study of low-frequency vibrations.

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